ANALYSIS OF A FORCED STRONGLY NON-LINEAR TWO-DEGREE-OF-FREEDOM SYSTEM BY MEANS OF THE POWER-SERIES METHOD

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A power-series solution is presented for the periodic motion of an undamped strongly non-linear two-degree-of-freedom system excited by harmonic forces. The analysis is facilitated by transforming the time variable into an harmonically oscillating time. The frequency of the new time variable is determined by observing the equality between the rate of change of dynamical energy and the power delivered by the forces. The results show good agreement with the modified Lindstedt–Poincaré method and the incremental harmonic balance method.

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1. INTRODUCTION

Classical perturbation techniques such as the Lindstedt–Poincaré method, the multiple-scales method [1], and the Krylov–Bogoluibov–Mitropolsky technique [2, 3] with its equivalent Galerkin method have commonly been restricted to the analysis of weakly non-linear oscillators. For strongly non-linear oscillators with multiple degrees-of-freedom, such methods are not sufficiently accurate, especially when internal resonance is involved, as demonstrated by Chen and Cheung [4]. Consequently, some modifications to existing procedures have been proposed. Geer and Andersen [5] used a hybrid perturbation-Galerkin technique which employs a perturbation expansion to give an approximate solution that is then used for a subsequent Galerkin analysis. Burton [6, 7] used the Lindstedt–Poincaré (LP) method and defined an expansion parameter to enable an accurate low order solution to be obtained for oscillators with odd non-linearity. More recently, Cheung *et al.* [8] proposed a modified LP method (MLP) by defining a new expansion parameter which remains small even if the original parameter grows without bound. In a subsequent paper [4], the MLP method was generalized to multiple-degree-of-freedom systems and applied to a clamped–hinged beam subjected to harmonic forces and discretized using a two-mode approximation.

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In this paper, a power-series solution is presented for the forced two-degree-of-freedom oscillator treated in reference [4]. The periodic motions of the oscillator are captured by transforming the time variable into an oscillating time which transforms the governing differential equations into a form solvable by the power-series method.

2. FORMULATION

Consider the vibrations of the harmonically forced two-degree-of-freedom system described by the equations:

$$\ddot{x} + x + Ax^3 + Bx^2y + Cxy^2 + Dy^3 = P_1 \cos \Omega_f t, \tag{1}$$

$$\ddot{y} + 9y + Ex^3 + Fx^2y + Gxy^2 + Hy^3 = P_2 \cos \Omega_f t \tag{2}$$

subject to the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \qquad y(0) = y_0, \qquad \dot{y}(0) = \dot{y}_0. \tag{3}$$

The overdot denotes differentiation with respect to time t. This discretized system is a result of a two-mode approximation of the large amplitude transverse vibration of an undamped clamped–hinged beam subjected to harmonic forces [4]. The non-linear elastic constants in equations (1) and (2) have the dimension of force per cubic length and assume the following values: $A = 0.2788$, $B = -0.3111$, $C = 1.116$, $D = -0.3864$, $E = B/3$, $F = C$, $G = 3D$, and $H = 3.8703$. The harmonic forces have amplitudes P_1, P_2 and a frequency Ω_f. Equations (1) and (2) are coupled non-linear differential equations whose solutions are, in general, non-periodic. Only under certain combinations of problem parameters and initial conditions does the solution become periodic. Because of the absence of an exact solution, the conditions under which periodic motion exists remain unknown and can only be studied approximately. The periodic solutions can be captured by transforming the time variable t into an harmonically oscillating time τ given by

$$\tau = \sin(\omega t), \tag{4}$$

whereby the infinite time t domain $(0 \leqslant t < \infty)$ is reduced to a finite time τ that oscillates between -1 and 1 at a frequency ω to be determined. Upon introducing equation (4) into equations (1–3), the transformed equations of motion become

$$\omega^2(1 - \tau^2)x'' - \omega^2\tau x' + x + Ax^3 + Bx^2y + Cxy^2 + Dy^3 = P_1 \cos \Omega_f t, \tag{5}$$

$$\omega^2(1 - \tau^2)y'' - \omega^2\tau y' + 9y + Ex^3 + Fx^2y + Gxy^2 + Hy^3 = P_2 \cos \Omega_f t \tag{6}$$

subject to initial conditions

$$x(0) = x_0, \qquad x'(0) = \dot{x}_0/\omega, \qquad y(0) = y_0, \qquad y'(0) = \dot{y}_0/\omega. \tag{7}$$

The prime denotes differentiation with respect to time τ. The above transformation allows power-series expansion of x and y in terms of τ. According to the theory of differential equations [9], equations (5) and (6) have one ordinary point at $\tau = 0$ and two regular singular points at $\tau = \pm 1$. It is convenient to expand x and y about the ordinary point as

$$x(\tau) = a_1 + a_2\tau + a_3\tau^2 + \cdots = \sum_{k=1}^{\infty} a_k \tau^{k-1}, \tag{8}$$

$$y(\tau) = b_1 + b_2\tau + b_3\tau^2 + \cdots = \sum_{k=1}^{\infty} b_k \tau^{k-1}, \tag{9}$$

where a_i and b_i are constant coefficients to be determined. Physically, the power-series solution means that the motion can be approximated by a finite number of series terms with the accuracy being ensured by including a sufficient number of such terms. Since τ is periodic, equations (8) and (9) are capable of capturing periodic motion which is conveniently assumed to start from the maximum displacement position. Under this condition, all the terms having odd powers of τ in equations (8) and (9) vanish and consequently, the same motion is repeated every half-cycle (positive or negative) of the oscillating time. It follows that the oscillating time frequency equals one-half the vibration frequency Ω, thus

$$\omega = \Omega/2. \tag{10}$$

The non-linear terms in equations (5) and (6) may, by different multiplications of equations (8) and (9), be expanded also as power series as follows:

$$x^3 = \sum_{k=1}^{\infty} c_k \tau^{k-1}, \quad x^2 y = \sum_{k=1}^{\infty} d_k \tau^{k-1}, \quad xy^2 = \sum_{k=1}^{\infty} e_k \tau^{k-1}, \quad y^3 = \sum_{k=1}^{\infty} f_k \tau^{k-1} \tag{11}$$

in which the constants c_k, d_k, e_k, and f_k can be computed once the constants $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ are known.

It is also necessary to express the forcing function $z = \cos(\Omega_f t)$ as a power series of τ. It can be verified that z satisfies the linear differential equation

$$(1 - \tau^2)z'' - \tau z' + r^2 z = 0, \tag{12}$$

where $r = \Omega_f/\omega$. It is interesting to note the similarity of the first terms of equations (5), (6), and (12). The solution to equation (12) can similarly be expanded about the ordinary point $\tau = 0$ as

$$z = \cos(\Omega_f t) = q_1 + q_2 \tau + q_3 \tau^2 + \cdots = \sum_{k=1}^{\infty} q_k \tau^{k-1}, \tag{13}$$

where q_i are constant coefficients which can be determined by substituting equation (13) into equation (12) giving

$$(1 - \tau^2) \sum_{k=1}^{\infty} (k-1)(k-2)q_k \tau^{k-3} - \sum_{k=1}^{\infty} (k-1)q_k \tau^{k-1} + r^2 \sum_{k=1}^{\infty} q_k \tau^{k-1} = 0. \tag{14}$$

By introducing a shift of index in the first term so that all terms have the same power, equation (14) can be rearranged as

$$\sum_{k=1}^{\infty} [k(k+1)q_{k+2} - (k-1)^2 q_k + r^2 q_k] \tau^{k-1} = 0. \tag{15}$$

If equation (12) is to be satisfied exactly, the coefficient of each power in equation (15) must identically vanish. This condition introduces the recurrence relation

$$q_{k+2} = \left[\frac{(k-1)^2 - r^2}{k(k+1)} \right] q_k, \qquad k = 1, 2, \ldots \tag{16}$$

between the series constants. The first two constants q_1 and q_2 are determined by requiring that equation (13) and its first derivative be satisfied at $t = 0$. This leads to $q_1 = 1$ and $q_2 = 0$. The remaining constants can be computed from equation (16) for a specified frequency ratio r. It may be of interest to note that for integer values of r, equation (13) reduces to the well-known Chebyshev polynomials.

Having represented the various terms in equations (5) and (6) by power series of τ, equations (8), (9), (11), and (13) can be substituted into equation (5) which, by following the same steps described above, becomes

$$\sum_{k=1}^{\infty} \{\omega^2[k(k+1)a_{k+2} - (k-1)^2 a_k] + a_k + Ac_k + Bd_k + Ce_k + Df_k - P_1 q_k\}\tau^{k-1} = 0. \quad (17)$$

Imposing the requirement of vanishing power coefficients in equation (17) leads to the recurrence relation

$$a_{k+2} = \frac{[(k-1)^2\omega^2 - 1]a_k - Ac_k - Bd_k - Ce_k - Df_k + P_1 q_k}{k(k+1)\omega^2}, \quad k = 1, 2, \ldots. \quad (18)$$

Similar substitution in equation (6) leads to the recurrence relation

$$b_{k+2} = \frac{[(k-1)^2\omega^2 - 9]b_k - Ec_k - Fd_k - Ge_k - Hf_k + P_2 q_k}{k(k+1)\omega^2}, \quad k = 1, 2, \ldots. \quad (19)$$

The starting values for recurrence relations (18) and (19) are obtained by introducing the initial conditions, equation (7), into equations (8) and (9) and assuming the motion starts from the maximum displacement position with zero velocities. This gives

$$a_1 = x_0, \qquad a_2 = 0, \qquad b_1 = y_0, \qquad b_2 = 0. \quad (20)$$

The remaining coefficients a_i and b_i depend recursively on these four fundamental constants and on the oscillating time frequency ω in accordance with equations (18) and (19). It follows that a solution, as expressed by equations (8) and (9), is obtained when the actual value of ω is determined. For that purpose, an auxiliary condition is invoked which expresses the equality between the rate of change of dynamical energy and the power delivered by the external forces. This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = P_1(\cos \Omega_f t)\dot{x} + P_2(\cos \Omega_f t)\dot{y}, \quad (21)$$

where T and V are the kinetic and potential energies for the system respectively. Equation (21) is chosen to solve for the oscillating time frequency ω and then, from equation (10), the vibration frequency Ω is twice that value. Multiplying equation (21) by $\mathrm{d}t/\mathrm{d}\tau$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(T+V) = \cos \Omega_f t(P_1 x' + P_2 y') = R_1 + R_2\tau + R_3\tau^2 + \cdots, \quad (22)$$

whereby equations (8), (9), and (13) are used to express the right-hand side as a single power series of τ. For the system under consideration, the potential energy V is obtained from its relations with the elastic forces $F_x = \partial V/\partial x$ and $F_y = \partial V/\partial y$ in equations (1) and (2), respectively, giving

$$V = \tfrac{1}{2}(x^2 + 9y^2) + \tfrac{1}{4}(Ax^4 + Hy^4) + \frac{B}{3}x^3 y + \frac{C}{2}x^2 y^2 + Dxy^3. \quad (23)$$

The kinetic energy of the system can be written as

$$T = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 = \tfrac{1}{2}\omega^2(1 - \tau^2)(x'^2 + y'^2). \quad (24)$$

Equation (22) may be integrated with respect to τ from $\tau = 0$ to 1 to give the characteristic equation

$$(T + V)_{\tau=1} - (T + V)_{\tau=0} - \left(R_1 + \frac{R_2}{2} + \frac{R_3}{3} + \cdots \right) = 0, \tag{25}$$

which may be solved for the actual oscillating time frequency by a frequency search. It is noted here that the kinetic energy is zero both at the start of motion ($\tau = 0$) because of the vanishing initial velocities and at $\tau = 1$ as given by equation (24). The potential energy is evaluated at the integration limits by using equations (8) and (9).

3. RESULTS AND DISCUSSION

The forced frequency response of the system governed by equations (1) and (2) was obtained by using the recurrence relations (18) and (19) in conjunction with the characteristic equation (25). Figure 1 compares the forced frequency response ($x_0 - \Omega_f$) obtained by the present method for $P_1 = 1$ and $P_2 = 0$ with those produced by available solutions [4]. It can be seen that the present method is in good agreement with the modified Linstedt–Poincaré (MLP) method and the incremental harmonic balance (IHB) method. The accuracy of the classical LP method deteriorates at large amplitudes. The power-series solution was obtained for a specified forcing frequency Ω_f and it was found that each branch of the response curve is associated with a specific value of the frequency ratio r. For example, branches (1) and (2) in Figure 1 are associated with $r = 3/2$ for which $\omega = (2/3) \Omega_f$ and the vibration frequency $\Omega = 2\omega = (4/3)\Omega_f$. On the other hand, branch (3) of the out-of-phase curve has a value $r = 2$ giving $\omega = 0·5\Omega_f$ and $\Omega = \Omega_f$. Consequently, each branch was constructed by specifying the oscillating time frequency and the characteristic
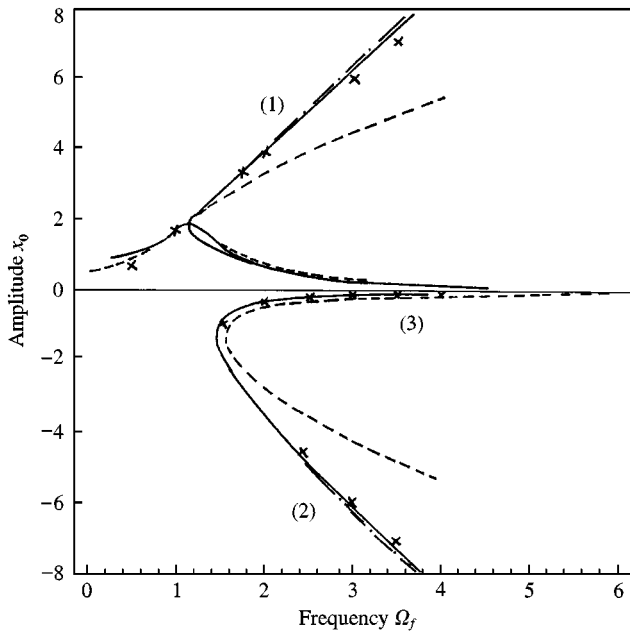

Figure 1. Forced frequency response $x_0 - \Omega_f$ fo $P_1 = 1$, $P_2 = 0$; ——, MLP method; – – – – –, LP method; –·–·–, IHB; ×, present.
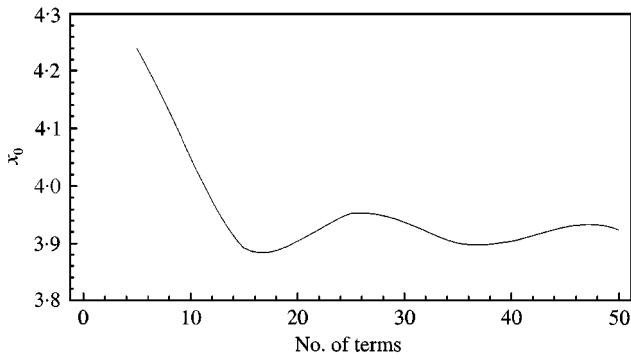
Figure 2. Convergence of amplitude x_0 for $P_1 = 1$, $P_2 = 0$; $\Omega_f = 2$, $r = 3/2$.

TABLE 1

Odd power-series coefficients ($P_1 = 1$, $P_2 = 0$, $\Omega_f = 3$, $r = 3/2$, $x_0 = 5\cdot97$)

	$i = 1$	$i = 3$	$i = 5$	$i = 7$	$i = 9$
a_i	5·9700	− 8·0361	3·0935	− 3·2872	2·1183
a_{i+10}	− 2·2849	1·7548	− 1·8526	1·5344	− 1·5995
a_{i+20}	1·3814	− 1·4287	1·2663	− 1·3022	1·1765
a_{i+30}	− 1·2038	1·1017	− 1·1246	1·0392	− 1·0576
b_i	0·000	2·7581	− 3·7406	3·7115	− 3·7378
b_{i+10}	3·6408	− 3·6631	3·5946	− 3·6043	3·5479
b_{i+20}	− 3·5476	3·4972	− 3·4914	3·4469	− 3·4347
b_{i+30}	3·3931	− 3·3798	3·3392	− 3·3239	3·2866

equation (25) was solved not for the frequency but for the amplitude x_0 by amplitude search. The value of y_0 was taken zero.

A convergence test was made for the power-series solution. Figure 2 shows the convergence of the amplitude x_0 for $P_1 = 1$, $P_2 = 0$, $\Omega_f = 2$, and $r = 3/2$ as the number of terms is increased. The results presented in this work were obtained using 40 terms. For smaller amplitudes, fewer number of terms is required to obtain accurate solutions. Table 1 shows the odd power-series coefficients for $P_1 = 1$, $P_2 = 0$, $\Omega_f = 3$, and $r = 3/2$ and $x_0 = 5\cdot97$ obtained with 40 terms. A progressive decrease in the absolute value of the coefficient is seen which characterizes a convergent solution. The even power coefficients were all zero because of the vanishing of initial velocities.

Figure 3 shows the forced response y_0-Ω_f for $P_1 = 1$, $P_2 = 0$, and $x_0 = 0$. Both in-phase and out-of-phase curves are associated with a frequency ratio $r = 2/3$ for which $\omega = (3/2)\Omega_f$ and $\Omega = 3\Omega_f$. Figure 4 shows the second fundamental resonance curves corresponding to $P_1 = 0$, $P_2 = 1$, and $x_0 = 0$. These curves were obtained with a value $r = 2$.

4. CONCLUSION

A power-series solution has been presented for the large amplitude periodic motion of an undamped two-degree-of-freedom system subjected to harmonic forces. The periodic motions were captured by transforming the time variable into an harmonically oscillating
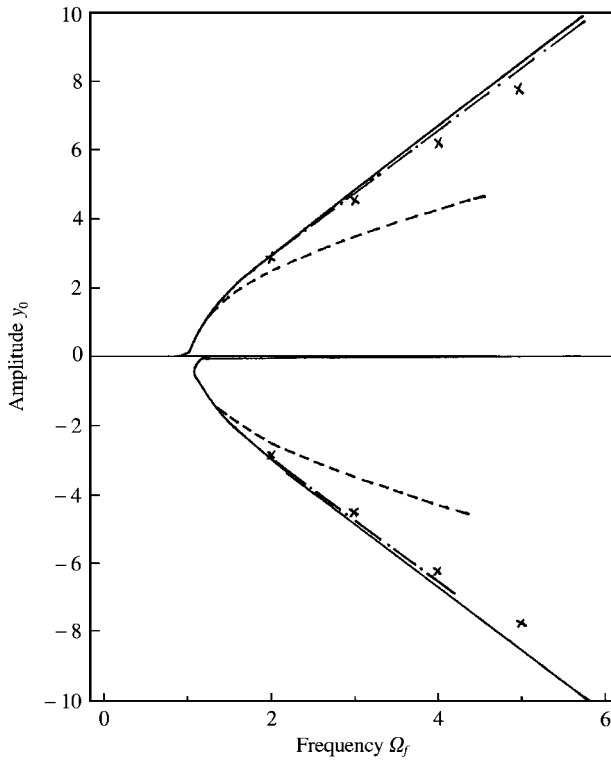
Figure 3. Forced frequency response $y_0 - \Omega_f$ for $P_1 = 1$, $P_2 = 0$; ———, MLP method; – – – – –, LP method; – · – · –, IHB; ×, present.
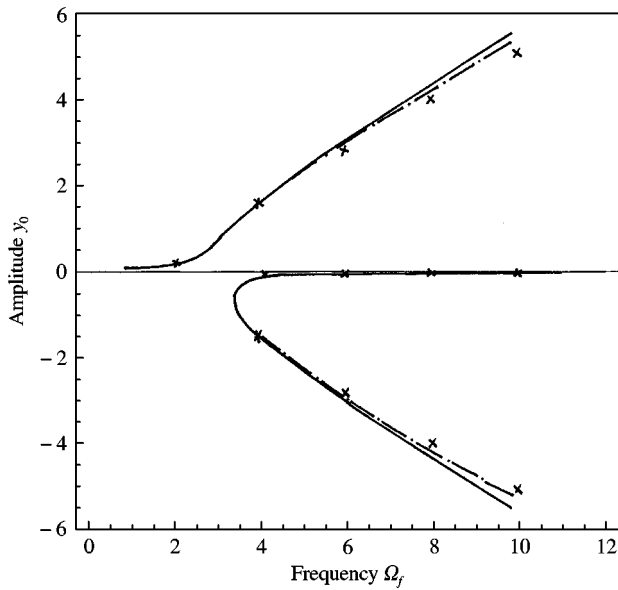

Figure 4. Forced frequency response $y_0 - \Omega_f$ for $P_1 = 0$, $P_2 = 1$: ———, MLP method; – – – – –, LP method; – · – · –, IHB; ×, present.

time. The results show good agreement with the modified Lindstedt–Poincaré method and the incremental harmonic balance method. A significant advantage of the present technique is its simple programmability and reduced computational effort. The solution can be applied to undamped strongly non-linear oscillators having multiple degrees-of-freedom and excited by harmonic forces. Oscillators with viscous damping and subjected to non-harmonic forces require a separate treatment.

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